

PRODUCTS OF COMMUTATORS IN A LIE NILPOTENT ASSOCIATIVE ALGEBRA. II

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ABSTRACT. Let F be a field of characteristic $\neq 2, 3$ and let A be a unital associative F -algebra. Define a left-normed commutator $[a_1, a_2, \dots, a_n]$ ($a_i \in A$) recursively by $[a_1, a_2] = a_1 a_2 - a_2 a_1$, $[a_1, \dots, a_{n-1}, a_n] = [[a_1, \dots, a_{n-1}], a_n]$ ($n \geq 3$). For $n \geq 2$, let $T^{(n)}(A)$ be the two-sided ideal in A generated by all commutators $[a_1, a_2, \dots, a_n]$ ($a_i \in A$). Define $T^{(1)}(A) = A$.

Let k, ℓ be integers such that $k > 0$, $0 \leq \ell \leq k$. Let m_1, \dots, m_k be positive integers such that ℓ of them are odd and $k - \ell$ of them are even. Let $N_{k\ell} = \sum_{i=1}^k m_i - 2k + \ell + 2$.

The aim of the present note is to show that, for any positive integers m_1, \dots, m_k , in general,

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \not\subseteq T^{(N_{k\ell}+1)}(A).$$

It is known that if $\ell < k$ (that is, if at least one of m_i is even) then, for each A ,

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_{k\ell})}(A)$$

so our result cannot be improved if $\ell < k$.

Let $N_k = \sum_{i=1}^k m_i - k + 1$. Recently Dangovski has proved that if m_1, \dots, m_k are any positive integers then, in general,

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \not\subseteq T^{(N_k+1)}(A).$$

Since $N_{k\ell} = N_k - (k - \ell - 1)$, Dangovski's result is stronger than ours if $\ell = k$ and is weaker than ours if $\ell \leq k - 2$; if $\ell = k - 1$ then $N_k = N_{k(k-1)}$ so both results coincide. It is known that if $\ell = k$ (that is, if all m_i are odd) then, for each A ,

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_k)}(A)$$

so in this case Dangovski's result cannot be improved.

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1. INTRODUCTION

Let R be an arbitrary associative and commutative unital ring and let A be a unital associative algebra over R . Define a left-normed commutator $[a_1, a_2, \dots, a_n]$ ($a_i \in A$) recursively by $[a_1, a_2] = a_1 a_2 - a_2 a_1$, $[a_1, \dots, a_{n-1}, a_n] = [[a_1, \dots, a_{n-1}], a_n]$ ($n \geq 3$). For $n \geq 2$, let $T^{(n)}(A)$ be the two-sided ideal in A generated by all commutators $[a_1, a_2, \dots, a_n]$ ($a_i \in A$). Define $T^{(1)}(A) = A$. Clearly, we have

$$A = T^{(1)}(A) \supseteq T^{(2)}(A) \supseteq T^{(3)}(A) \supseteq \dots \supseteq T^{(n)}(A) \supseteq \dots$$

We are concerned with the following.

Problem 1. Let $k \geq 2$ and let m_1, \dots, m_k be positive integers. Find the maximal integer $N = N(R, m_1, \dots, m_k)$ such that, for each R -algebra A ,

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N)}(A).$$

Let $X = \{x_1, x_2, \dots\}$ be an infinite countable set and let $R\langle X \rangle$ be the free associative algebra over R freely generated by X . Define $T^{(n)} = T^{(n)}(R\langle X \rangle)$.

Problem 2. Let $k \geq 2$ and let m_1, \dots, m_k be positive integers. Find the maximal integer $N = N(R, m_1, \dots, m_k)$ such that

$$T^{(m_1)} \dots T^{(m_k)} \subseteq T^{(N)}.$$

It is easy to check that Problem 1 is equivalent to Problem 2, and the integer N in both problems is the same.

Problem 2 and some other similar questions have been recently studied by Dangovski [4] (using different terminology). The work of Dangovski was motivated by the results of Etingof, Kim and Ma [7] and Bapat and Jordan [2], which in turn were motivated by the pioneering article by Feigin and Shoikhet [8].

The following assertion was proved by Latyshev [15, Lemma 1] in 1965 (Latyshev's paper was published in Russian) and independently rediscovered by Gupta and Levin [13, Theorem 3.2] in 1983.

Theorem 1.1 (see [13, 15]). *Let R be an arbitrary associative and commutative unital ring and let A be an associative R -algebra. Let $m, n \in \mathbb{Z}$, $m, n \geq 1$. Then*

$$T^{(m)}(A) T^{(n)}(A) \subseteq T^{(m+n-2)}(A).$$

Latyshev [15] has actually proved that $T^{(m)} T^{(n)} \subseteq T^{(m+n-2)}$ in $R\langle X \rangle$; this assertion is equivalent to Theorem 1.1.

Note that, for a unital associative ring R , we have $\frac{1}{6} \in R$ if and only if $2(= 1 + 1)$ and 3 are invertible in R . The theorem below was proved by Sharma and Srivastava [17, Theorem 2.8] in 1990 and independently rediscovered (with different proofs) by Bapat and Jordan [2, Corollary 1.4] in 2013 and by Grishin and Pchelintsev [10, Theorem 1] in 2015.

Theorem 1.2 (see [2, 10, 17]). *Let R be an arbitrary associative and commutative unital ring such that $\frac{1}{6} \in R$ and let A be an associative R -algebra. Let $m, n \in \mathbb{Z}$, $m, n > 1$ and at least one of the numbers m, n is odd. Then*

$$T^{(m)}(A) T^{(n)}(A) \subseteq T^{(m+n-1)}(A).$$

Note that Grishin and Pchelintsev [10] have actually proved that $T^{(m)} T^{(n)} \subseteq T^{(m+n-1)}$; this result is equivalent to Theorem 1.2.

Let $N_k = \sum_{i=1}^k m_i - k + 1$. The proposition below follows immediately from Theorem 1.2.

Proposition 1.3. *Let R be an arbitrary associative and commutative unital ring such that $\frac{1}{6} \in R$ and let A be an associative R -algebra. Let $k > 0$ be an integer and let $m_i > 0$ ($i = 1, \dots, k$) be **odd** integers. Then*

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_k)}(A).$$

Let $N_{k\ell} = \sum_{i=1}^k m_i - 2k + \ell + 2 = N_k - (k - \ell - 1)$. One can deduce from Theorems 1.1 and 1.2 the following proposition (see Dangovski [4, Section 6]).

Proposition 1.4 (see [4]). *Let R be an arbitrary associative and commutative unital ring such that $\frac{1}{6} \in R$ and let A be an associative R -algebra. Let k, ℓ be integers such that $0 \leq \ell < k$. Let $m_i \geq 2$ ($i = 1, \dots, k$) be integers such that ℓ of them are odd and $(k - \ell) > 0$ of them are even. Then*

$$(1) \quad T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_{k\ell})}(A).$$

We prove Proposition 1.4 in Section 2 in order to have the paper more self-contained.

Recently Dangovski [4, Proposition 2.2] has proved a result that can be reformulated as follows.

Theorem 1.5 (see [4]). *Let F be a field and let k be a positive integer. Let m_1, \dots, m_k be positive integers and let N_k be as above. Then there exists an associative F -algebra A such that*

$$(2) \quad T^{(m_1)}(A) \dots T^{(m_k)}(A) \not\subseteq T^{(N_k+1)}(A).$$

One can deduce from Theorem 1.5 the following.

Corollary 1.6. *Let R be an arbitrary unital associative and commutative ring and let k, m_1, \dots, m_k, N_k be as in Theorem 1.5. Then there exists an associative R -algebra A such that (2) holds.*

Proof. Suppose that R is not a field. Let M be a maximal ideal of R (by Zorn's lemma, such an ideal M exists). Then $F = R/M$ is a field and the F -algebra A of Theorem 1.5 can be viewed in a natural way as an R -algebra (with $r \cdot a$ defined by $r \cdot a = (r + M) \cdot a$ for $r \in R, a \in A$). Since A satisfies (2), the result follows. \square

Let N be the integer defined in Problems 1 and 2. If $\frac{1}{6} \in R$ and all the integers m_1, \dots, m_k are odd then $N = N_k$. Indeed, it follows from Proposition 1.3 and Corollary 1.6 that in this case we always have

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subset T^{(N_k)}(A)$$

and, in general,

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \not\subset T^{(N_k+1)}(A).$$

Suppose that ℓ of the integers m_1, \dots, m_k are odd ($\ell < k$) and $(k - \ell) > 0$ of them are even. Let $\frac{1}{6} \in R$. Then, by Proposition 1.4, $N_{k\ell} \leq N$ and, by Corollary 1.6, $N \leq N_k$. If $\ell = k - 1$ (that is, $k - 1$ of the integers m_1, \dots, m_k are odd and one of them is even) then $N_{k(k-1)} = N_k$ so $N = N_k$. However, if $0 \leq \ell < k - 1$ then $N_{k\ell} = N_k - (k - \ell - 1) < N_k$ so one can only deduce from the results above that $N_{k\ell} \leq N \leq N_k$.

Our main result is as follows.

Theorem 1.7. *Let F be a field. Let k, ℓ be integers, $0 \leq \ell \leq k$. Let m_1, \dots, m_k be positive integers such that ℓ of them are odd and $k - \ell$ of them are even and let $N_{k\ell}$ be as above. Then there exists a unital associative F -algebra A such that*

$$(3) \quad T^{(m_1)}(A) \dots T^{(m_k)}(A) \not\subset T^{(N_{k\ell}+1)}(A).$$

In a particular case when $k = 2$ and m_1, m_2 are even Theorem 1.7 has been recently proved by Grishin and Pchelintsev [10] and independently by the authors of the present article [6]. In another particular case when $m_1 = m_2 = \dots = m_{k-1} = 2$ and m_k is even this theorem has been proved by Grishin, Tsybulya and Shokola [11, Theorem 3].

The proof of the following result is similar to that of Corollary 1.6.

Corollary 1.8. *Let R be an arbitrary unital associative and commutative ring and let $k, \ell, m_1, \dots, m_k, N_{k\ell}$ be as in Theorem 1.7. Then there exists an associative R -algebra A such that (3) holds.*

It follows that if $\frac{1}{6} \in R$ and at least one of the integers m_i is even then $N = N_{k\ell}$ because, by Proposition 1.4 and Corollary 1.8, in this case we always have

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_{k\ell})}(A)$$

but, in general,

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \not\subset T^{(N_{k\ell}+1)}(A).$$

Thus, the solution of Problems 1 and 2 (for R that contains $\frac{1}{6}$) is as follows. Let R be a unital associative and commutative ring such that $\frac{1}{6} \in R$ and let k, m_1, \dots, m_k be positive integers. Then

$$N = \begin{cases} N_k = \sum_{i=1}^k m_i - k + 1 & \text{if all integers } m_i \text{ are odd (Dangovski [4]);} \\ N_{k\ell} = \sum_{i=1}^k m_i - 2k + \ell + 2 & \text{if } \ell < k \text{ of the integers } m_i \text{ are odd and } k - \ell \text{ of them are even.} \end{cases}$$

Recall that an associative algebra A is Lie nilpotent of class at most c if $[u_1, \dots, u_c, u_{c+1}] = 0$ for all $u_i \in A$. Theorem 1.7 follows immediately from the following result.

Theorem 1.9. *Under the hypotheses of Theorem 1.7, there exists a unital associative F -algebra A such that the following two conditions are satisfied:*

- i) $T^{(N_{k\ell}+1)}(A) = 0$, that is, the algebra A is Lie nilpotent of class at most $N_{k\ell}$;
- ii) there are $v_{ij} \in A$ such that

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] \neq 0.$$

To prove Theorem 1.9 we use the same algebra A that was used in [6, Theorem 1.4].

Remarks. 1. Both Theorem 1.5 and Theorem 1.7 are valid for arbitrary k -tuples m_1, m_2, \dots, m_k of positive integers. However, if $\ell = k$ (that is, if all m_i are odd) then Theorem 1.5 gives a stronger result than Theorem 1.7 because $N_{kk} = N_k + 1 > N_k$ and therefore $T^{(N_{kk}+1)}(A) \subset T^{(N_k+1)}(A)$. If $\ell = k - 1$ (that is, if one of the integers m_1, m_2, \dots, m_k is even and $k - 1$ of them are odd) then $N_{k(k-1)} = N_k$ so the results of Theorem 1.5 and Theorem 1.7 coincide; and if $\ell < k - 1$ (that is, if two or more of the integers m_1, m_2, \dots, m_k are even) then $N_{k\ell} = N_k - (k - \ell - 1) < N_k$ so Theorem 1.5 gives a weaker result than Theorem 1.7.

2. The proofs of Theorem 1.2 given in [2], [10] and [17] are valid for algebras over an associative and commutative unital ring R such that $\frac{1}{6} \in R$. However, the proof given in [2] can be slightly modified to become also valid over any R such that $\frac{1}{3} \in R$ (see [1, Remark 3.9] for explanation). Moreover, for some specific m and n Theorem 1.2 holds over an arbitrary ring R : for instance, $T^{(3)}(A)T^{(3)}(A) \subset T^{(5)}(A)$ for any algebra A over any associative and commutative unital ring R (see [3, Lemma 2.1]). However, in general Theorem 1.2 fails over \mathbb{Z} and over a field of characteristic 3: it was shown in [5, 14] that in this case $T^{(3)}T^{(2)} \not\subset T^{(4)}$ and moreover, $T^{(3)}(T^{(2)})^\ell \not\subset T^{(4)}$ for all $\ell \geq 1$.

3. In 1978 Volichenko proved Theorem 1.2 for $m = 3$ and arbitrary n in the preprint [18] written in Russian. In 1985 Levin and Sehgal [16] independently rediscovered Volichenko's result. More recently Etingof, Kim and Ma [7] and Gordienko [9] have independently proved this theorem for small m and n ; these authors were unaware of the results of [16, 18].

2. PROOFS OF PROPOSITION 1.4 AND THEOREM 1.9

Proof of Proposition 1.4. Induction on k . If $k = 1$ then $\ell = 0$ so $N_{10} = m_1$ and (1) holds.

Suppose that $k > 1$ and for all products of less than k terms $T^{(m_i)}(A)$ the proposition has already been proved. We split the proof in 3 cases.

Case 1. Suppose that m_k is odd. Then for some i such that $1 \leq i < k$ the number m_i is even so we can apply the induction hypothesis to the product $T^{(m_1)}(A) \dots T^{(m_{k-1})}(A)$. By this hypothesis,

$$T^{(m_1)}(A) \dots T^{(m_{k-1})}(A) \subset T^{(N')}(A)$$

where $N' = \sum_{i=1}^{k-1} m_i - 2(k-1) + (\ell-1) + 2 = \sum_{i=1}^{k-1} m_i - 2k + \ell + 3$. By Theorem 1.2,

$$T^{(N')}(A) T^{(m_k)}(A) \subset T^{(N'+m_k-1)}(A) = T^{(N_{k\ell})}(A)$$

since $N' + m_k - 1 = \sum_{i=1}^k m_i - 2k + \ell + 2 = N_{k\ell}$. Thus, in this case (1) holds, as required.

Case 2. Suppose that m_k is even and, for some i such that $1 \leq i < k$, m_i is also even. Then we can apply the induction hypothesis to the product $T^{(m_1)}(A) \dots T^{(m_{k-1})}(A)$ so

$$T^{(m_1)}(A) \dots T^{(m_{k-1})}(A) \subset T^{(N'')}(A)$$

where $N'' = \sum_{i=1}^{k-1} m_i - 2(k-1) + \ell + 2 = \sum_{i=1}^{k-1} m_i - 2k + \ell + 4$. By Theorem 1.1,

$$T^{(N'')}(A) T^{(m_k)}(A) \subset T^{(N''+m_k-2)}(A) = T^{(N_{k\ell})}(A)$$

since $N'' + m_k - 2 = \sum_{i=1}^k m_i - 2k + \ell + 2 = N_{k\ell}$. Hence, in this case (1) holds, as required.

Case 3. Suppose that m_k is even and all m_i for $1 \leq i < k$ are odd. Applying Theorem 1.2 $k-1$ times, we get

$$T^{(m_1)}(A) \dots T^{(m_{k-1})}(A) T^{(m_k)}(A) \subset T^{(m_1+\dots+m_k-k+1)}(A) = T^{(N_{k(k-1)})}(A)$$

since $\sum_{i=1}^k m_i - k + 1 = \sum_{i=1}^k m_i - 2k + (k-1) + 2 = N_{k(k-1)}$. Thus, in this case (1) also holds.

The proof of Proposition 1.4 is completed. \square

The proof of Theorem 1.9 below is a modification of the proof of [6, Theorem 1.4]. First we need some auxiliary results.

Let G and H be unital associative algebras over a field F such that $[g_1, g_2, g_3] = 0$, $[h_1, h_2, h_3] = 0$ for all $g_i \in G$, $h_j \in H$. Note that each commutator $[g_1, g_2]$ ($g_i \in G$) is central in G , that is,

$[g_1, g_2]g = g[g_1, g_2]$ for each $g \in G$. Similarly, each commutator $[h_1, h_2]$ ($h_j \in H$) is central in H . The following lemma has been proved in [6, Lemma 2.1] by induction on n .

Lemma 2.1 (see [6]). *Let*

$$c_\ell = [g_1 \otimes h_1, g_2 \otimes h_2, \dots, g_\ell \otimes h_\ell]$$

where $\ell \geq 2, g_i \in G, h_j \in H$. Then

$$\begin{aligned} c_2 &= [g_1, g_2] \otimes h_1 h_2 + g_2 g_1 \otimes [h_1, h_2], \\ c_{2n} &= [g_1, g_2][g_3, g_4] \dots [g_{2n-1}, g_{2n}] \otimes [h_1 h_2, h_3][h_4, h_5] \dots [h_{2n-2}, h_{2n-1}] h_{2n} \\ &\quad + [g_2 g_1, g_3][g_4, g_5] \dots [g_{2n-2}, g_{2n-1}] g_{2n} \otimes [h_1, h_2][h_3, h_4] \dots [h_{2n-1}, h_{2n}] \quad (n > 1), \\ c_{2n+1} &= [g_1, g_2][g_3, g_4] \dots [g_{2n-1}, g_{2n}] g_{2n+1} \otimes [h_1 h_2, h_3][h_4, h_5] \dots [h_{2n}, h_{2n+1}] \\ &\quad + [g_2 g_1, g_3][g_4, g_5] \dots [g_{2n}, g_{2n+1}] \otimes [h_1, h_2][h_3, h_4] \dots [h_{2n-1}, h_{2n}] h_{2n+1} \quad (n \geq 1). \end{aligned}$$

Corollary 2.2 (see [6]). *Suppose that*

$$(4) \quad [f_1, f_2] \dots [f_{2n-1}, f_{2n}] = 0 \quad \text{for all } f_j \in H.$$

Then for all $u_i \in G \otimes H$ we have

$$[u_1, u_2, \dots, u_{2n+1}] = 0.$$

Proof. It follows from (4) and Lemma 2.1 that $[g_1 \otimes h_1, g_2 \otimes h_2, \dots, g_{2n+1} \otimes h_{2n+1}] = 0$ for all $g_i \in G, h_j \in H$. Since each $u_i \in G \otimes H$ is a sum of products of the form $g \otimes h$ ($g \in G, h \in H$), we have $[u_1, u_2, \dots, u_{2n+1}] = 0$ for all $u_i \in G \otimes H$, as required. \square

The following assertion follows immediately from Lemma 2.1.

Corollary 2.3. *Let $v_1 = g_1 \otimes 1, v_i = g_i \otimes h_i$ ($i = 2, \dots, 2m' - 1$), $v_{2m'} = g_{2m'} \otimes 1$ and let $w_1 = g'_1 \otimes 1, w_j = g'_j \otimes h'_j$ ($j = 2, \dots, 2n' + 1$) where $g_i, g'_i \in G, h_j, h'_j \in H$. Then*

$$\begin{aligned} [v_1, \dots, v_{2m'}] &= [g_1, g_2] \dots [g_{2m'-1}, g_{2m'}] \otimes [h_2, h_3] \dots [h_{2m'-2}, h_{2m'-1}], \\ [w_1, \dots, w_{2n'+1}] &= [g'_1, g'_2] \dots [g'_{2n'-1}, g'_{2n'}] g'_{2n'+1} \otimes [h'_2, h'_3] \dots [h'_{2n'}, h'_{2n'+1}]. \end{aligned}$$

Proof of Theorem 1.9. Two cases are to be considered: the case when $\text{char } F \neq 2$ and the case when $\text{char } F = 2$.

Case 1. Suppose that F is a field of characteristic $\neq 2$. Let E be the unital infinite-dimensional Grassmann (or exterior) algebra over F . Then E is generated by the elements e_i ($i = 1, 2, \dots$) such that $e_i e_j = -e_j e_i, e_i^2 = 0$ for all i, j and the set

$$\mathcal{B} = \{e_{i_1} e_{i_2} \dots e_{i_k} \mid k \geq 0, i_1 < i_2 < \dots < i_k\}$$

forms a basis of E over F . It is well known and easy to check that $[g_1, g_2, g_3] = 0$ for all $g_i \in E$.

Recall that the r -generated unital Grassmann algebra E_r is the unital subalgebra of E generated by e_1, e_2, \dots, e_r . Note that $[h_1, h_2, h_3] = 0$ for all $h_j \in E_r$.

Take $A = E \otimes E_r$ where $r = \sum_{i=1}^k m_i - 2k + \ell = N_{k\ell} - 2$. It is easy to check that r is an even integer. We can apply Lemma 2.1 and Corollaries 2.2 and 2.3 for $G = E, H = E_r$.

Note that $[f_1, f_2] \dots [f_{r+1}, f_{r+2}] = 0$ for all $f_i \in E_r$. Indeed, for all $f, f' \in E_r$ the commutator $[f, f']$ belongs to the linear span of the set $\{e_{i_1} \dots e_{i_{2\ell}} \mid \ell \geq 1, 1 \leq i_s \leq r\}$. Hence, $[f_1, f_2] \dots [f_{r+1}, f_{r+2}]$ belongs to the linear span of the set $\{e_{i_1} \dots e_{i_{2\ell}} \mid \ell \geq (r+2)/2, 1 \leq i_s \leq r\}$. Since $2\ell \geq r+2 > r$, each product $e_{i_1} \dots e_{i_{2\ell}}$ above contains equal terms $e_{i_s} = e_{i_{s'}}$ ($s < s'$) and, therefore, is equal to 0. Thus, $[f_1, f_2] \dots [f_{r+1}, f_{r+2}] = 0$, as claimed.

Since $N_{k\ell} = r + 2$, we have $[f_1, f_2] \dots [f_{(N_{k\ell}-1)}, f_{N_{k\ell}}] = 0$ for all $f_i \in E_r$. Hence, by Corollary 2.2, we have $[u_1, \dots, u_{(N_{k\ell}+1)}] = 0$ for all $u_i \in A = E \otimes E_r$, that is, $T^{(N_{k\ell}+1)}(A) = 0$, as required.

Now it suffices to find elements $v_{ij} \in A$ such that

$$(5) \quad [v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] \neq 0.$$

Let

$$\mathcal{P} = \{(i, j) \mid 1 \leq i \leq k; 1 \leq j \leq m_i\}.$$

Note that v_{ij} appears in (5) if and only if $(i, j) \in \mathcal{P}$. Let $\mathcal{N} = \sum_{i=1}^k m_i$ and let $\mu : \mathcal{P} \rightarrow \{1, 2, \dots, \mathcal{N}\}$ be a bijection. Define

$$e_{ij} = e_{\mu(i,j)} \quad ((i, j) \in \mathcal{P}).$$

Note that

$$(6) \quad \prod_{(i,j) \in \mathcal{P}} e_{ij} = (-1)^\delta e_1 e_2 \dots e_{\mathcal{N}}$$

for some $\delta \in \{0, 1\}$.

Let $\mathcal{P}' \subset \mathcal{P}$,

$$\mathcal{P}' = \{(i', j') \mid 1 \leq i' \leq k; 2 \leq j' \leq m_i - 1 \text{ if } m_i \text{ is even; } 2 \leq j' \leq m_i \text{ if } m_i \text{ is odd}\}.$$

Let $\mu' : \mathcal{P}' \rightarrow \{1, 2, \dots, \sum_{i=1}^k m_i - 2k + \ell\} = \{1, 2, \dots, r\}$ be a bijection. Define

$$e'_{i'j'} = e_{\mu'(i',j')} \quad ((i', j') \in \mathcal{P}').$$

Note that

$$(7) \quad \prod_{(i',j') \in \mathcal{P}'} e'_{i'j'} = (-1)^{\delta'} e_1 e_2 \dots e_r$$

for some $\delta' \in \{0, 1\}$.

Define

$$\begin{aligned} v_{i1} &= e_{i1} \otimes 1; \\ v_{ij} &= e_{ij} \otimes e'_{ij} \quad (1 \leq i \leq k; 2 \leq j \leq m_i - 1); \\ v_{im_i} &= \begin{cases} e_{im_i} \otimes 1 & \text{if } m_i \text{ is even;} \\ e_{im_i} \otimes e'_{im_i} & \text{if } m_i \text{ is odd.} \end{cases} \end{aligned}$$

If m_i is even then, by Corollary 2.3,

$$[v_{i1}, v_{i2}, \dots, v_{im_i}] = [e_{i1}, e_{i2}][e_{i3}, e_{i4}] \dots [e_{i(m_i-1)}, e_{im_i}] \otimes [e'_{i2}, e'_{i3}][e'_{i4}, e'_{i5}] \dots [e'_{i(m_i-2)}, e'_{i(m_i-1)}].$$

Note that $e_{st}e_{s't'} = -e_{s't'}e_{st}$ for all s, s', t, t' so $[e_{st}, e_{s't'}] = 2e_{st}e_{s't'}$. It follows that if m_i is even then

$$[v_{i1}, v_{i2}, \dots, v_{im_i}] = 2^{m_i-1} e_{i1} e_{i2} \dots e_{im_i} \otimes e'_{i2} e'_{i3} \dots e'_{i(m_i-1)}.$$

If m_i is odd then, by Corollary 2.3,

$$\begin{aligned} [v_{i1}, v_{i2}, \dots, v_{im_i}] &= [e_{i1}, e_{i2}][e_{i3}, e_{i4}] \dots [e_{i(m_i-2)}, e_{i(m_i-1)}] e_{im_i} \otimes [e'_{i2}, e'_{i3}][e'_{i4}, e'_{i5}] \dots [e'_{i(m_i-1)}, e'_{im_i}] \\ &= 2^{m_i-1} e_{i1} e_{i2} \dots e_{i(m_i-1)} e_{im_i} \otimes e'_{i2} e'_{i3} \dots e'_{i(m_i-1)} e'_{im_i}. \end{aligned}$$

It follows that

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] = 2^{N_k-1} \prod_{i=1}^k \prod_{j=1}^{m_i} e_{ij} \otimes \prod_{i'=1}^k \prod_{j'=2}^{m'_{i'}} e'_{i'j'}$$

where

$$m'_{i'} = \begin{cases} m_{i'} - 1 & \text{if } m_{i'} \text{ is even;} \\ m_{i'} & \text{if } m_{i'} \text{ is odd,} \end{cases}$$

that is,

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] = 2^{N_k-1} \prod_{(i,j) \in \mathcal{P}} e_{ij} \otimes \prod_{(i',j') \in \mathcal{P}'} e'_{i'j'}.$$

By (6) and (7), we have

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] = (-1)^{\delta+\delta'} 2^{N_k-1} e_1 e_2 \dots e_{\mathcal{N}} \otimes e_1 e_2 \dots e_r \neq 0,$$

as required.

Case 2. Suppose that F is a field of characteristic 2. Let \mathcal{G} be the group given by the presentation

$$\mathcal{G} = \langle y_1, y_2, \dots \mid y_i^2 = 1, ((y_i, y_j), y_k) = 1 \ (i, j, k = 1, 2, \dots) \rangle$$

where $(a, b) = a^{-1}b^{-1}ab$. Then it is easy to check that \mathcal{G} is a nilpotent group of class 2 so $(a, b)c = c(a, b)$ for all $a, b, c \in \mathcal{G}$ and, therefore, $(a, bc) = (a, c)c^{-1}(a, b)c = (a, b)(a, c)$ (see [6] for more details). It is clear that the quotient group \mathcal{G}/\mathcal{G}' is an elementary abelian 2-group so $b^2 \in \mathcal{G}' \subseteq Z(\mathcal{G})$ for all $b \in \mathcal{G}$. It follows that $(a, b^2) = 1$ so $(a, b)^2 = (a, b^2) = 1$, that is, $(a, b) = (a, b)^{-1}$. Since $(b, a) = (a, b)^{-1}$, we have $(a, b) = (b, a)$ for all $a, b \in \mathcal{G}$.

Let $(<)$ be an arbitrary linear order on the set $\{(i, j) \mid i, j \in \mathbb{Z}, 0 < i < j\}$. The following lemma is well known and easy to check.

Lemma 2.4. *Let $a \in \mathcal{G}$. Then a can be written in a unique way in the form*

$$(8) \quad a = y_{i_1} \dots y_{i_q}(y_{j_1}, y_{j_2}) \dots (y_{j_{2q'-1}}, y_{j_{2q'}})$$

where $q, q' \geq 0$; $i_1 < \dots < i_q$, $j_{2s-1} < j_{2s}$ for all s , $(j_{2s-1}, j_{2s}) < (j_{2s'-1}, j_{2s'})$ if $s < s'$.

Let $F\mathcal{G}$ be the group algebra of \mathcal{G} over F . Let $d_{ij} = (y_i, y_j) + 1 \in F\mathcal{G}$. Note that $d_{ij} = d_{ji}$ and $d_{ii} = 0$ for all i, j .

Let I be the two-sided ideal of $F\mathcal{G}$ generated by the set

$$S = \{d_{i_1 i_2} d_{i_3 i_4} + d_{i_1 i_3} d_{i_2 i_4} \mid i_1, i_2, i_3, i_4 = 1, 2, \dots\}.$$

The following two lemmas are well known (see, for instance, [12, Lemma 2.1], [13, Example 3.8]); their proofs can also be found in [6].

Lemma 2.5. *For all $u_1, u_2, u_3 \in F\mathcal{G}$, we have $[u_1, u_2, u_3] \in I$.*

Lemma 2.6. *For all $\ell > 0$, we have $((y_1, y_2) + 1)((y_3, y_4) + 1) \dots ((y_{2\ell-1}, y_{2\ell}) + 1) \notin I$.*

Since the ideal I is invariant under all permutations of the set $\{y_1, y_2, \dots\}$ of generators of the group \mathcal{G} , we have the following.

Corollary 2.7. *Let $\ell > 0$. Then $((y_{i_1}, y_{i_2}) + 1) \dots ((y_{i_{2\ell-1}}, y_{i_{2\ell}}) + 1) \notin I$ if all integers $i_1, i_2, \dots, i_{2\ell}$ are distinct.*

Now we are in a position to complete the proof of Theorem 1.9. Recall that $r = \sum_{i=1}^k m_i - 2k + \ell = N_{k\ell} - 2$ is an even integer. Let \mathcal{G}_r be the subgroup of \mathcal{G} generated by y_1, \dots, y_r ; let $I_r = I \cap F\mathcal{G}_r$. Take $G = F\mathcal{G}/I$, $H = F\mathcal{G}_r/I_r$. Take $A = G \otimes H$. By Lemma 2.5, we can apply Lemma 2.1 and Corollaries 2.2 and 2.3 to A .

We claim that $[f_1, f_2] \dots [f_{r+1}, f_{r+2}] \in I_r$ for all $f_i \in F\mathcal{G}_r$. Indeed, we may assume without loss of generality that $f_i \in \mathcal{G}_r$ for all i . Since

$$[f_{2s-1}, f_{2s}] = f_{2s-1}f_{2s} + f_{2s}f_{2s-1} = f_{2s-1}f_{2s}((f_{2s}, f_{2s-1}) + 1) = f_{2s-1}f_{2s}((f_{2s-1}, f_{2s}) + 1)$$

(recall that F is a field of characteristic 2), we have

$$[f_1, f_2] \dots [f_{r+1}, f_{r+2}] = f_1 f_2 \dots f_{r+2}((f_1, f_2) + 1) \dots ((f_{r+1}, f_{r+2}) + 1).$$

It is clear that, for each s , $(f_{2s-1}, f_{2s}) = \prod_t c_{i_{st}j_{st}}$ for some commutators $c_{i_{st}j_{st}} = (y_{i_{st}}, y_{j_{st}})$. Let $d_{i_{st}j_{st}} = c_{i_{st}j_{st}} + 1$; then $c_{i_{st}j_{st}} = d_{i_{st}j_{st}} - 1$. We have

$$(f_{2s-1}, f_{2s}) + 1 = \prod_t c_{i_{st}j_{st}} + 1 = \left(\prod_t (d_{i_{st}j_{st}} - 1) \right) + 1 = \prod_t d_{i_{st}j_{st}} + \dots + \sum_{t < t'} d_{i_{st}j_{st}} d_{i_{st'}j_{st'}} + \sum_t d_{i_{st}j_{st}}.$$

It follows that the product $((f_1, f_2) + 1) \dots ((f_{r+1}, f_{r+2}) + 1)$ can be written as a sum of products of the form

$$(9) \quad d_{q_1 q_2} \dots d_{q_{2\ell-1} q_{2\ell}} = ((y_{q_1}, y_{q_2}) + 1) \dots ((y_{q_{2\ell-1}}, y_{q_{2\ell}}) + 1)$$

where $2\ell \geq r + 2 > r$. Hence, in the product (9) we have $q_t = q_{t'}$ for some $t < t'$.

Note that $d_{j_1 j_3} d_{j_2 j_3} \in I$ for all j_1, j_2, j_3 because $d_{j_1 j_3} d_{j_2 j_3} = d_{j_1 j_3} d_{j_2 j_3} + d_{j_1 j_2} d_{j_3 j_3} \in S$. Since $d_{ij} = d_{ji}$ for all i, j , we have $d_{i_1 i_2} d_{i_3 i_4} \in I$ if any two of the indices i_1, i_2, i_3, i_4 coincide. It follows that each

product (9) belongs to $I_r = I \cap F\mathcal{G}_r$ and so does the product $((f_1, f_2) + 1) \dots ((f_{r+1}, f_{r+2}) + 1)$. Hence, $[f_1, f_2] \dots [f_{r+1}, f_{r+2}] \in I_r$, as claimed. Since $N_{k\ell} = r + 2$, we have $[f_1, f_2] \dots [f_{(N_{k\ell}-1)}, f_{N_{k\ell}}] \in I_r$ for all $f_i \in F\mathcal{G}_r$.

For any $u \in F\mathcal{G}$, let $\bar{u} = u + I \in G = F\mathcal{G}/I$. Since one can view the algebra $H = F\mathcal{G}_r/I_r$ as a subalgebra of $G = F\mathcal{G}/I$, we also write $\bar{u} = u + I_r \in H = F\mathcal{G}_r/I_r$ for $u \in F\mathcal{G}_r$.

By the observation above, $[\bar{f}_1, \bar{f}_2] \dots [\bar{f}_{(N_{k\ell}-1)}, \bar{f}_{N_{k\ell}}] = 0$ for all $\bar{f}_i \in H$. Hence, by Corollary 2.2, we have $[u_1, \dots, u_{(N_{k\ell}+1)}] = 0$ for all $u_i \in A = G \otimes H$, that is, $T^{(N_{k\ell}+1)}(A) = 0$, as required.

Let \mathcal{P} , \mathcal{P}' , μ and μ' be as in Case 1. Recall that $\mathcal{N} = \sum_{i=1}^k m_i$. Define

$$y_{ij} = y_{\mu(i,j)} \quad ((i, j) \in \mathcal{P}), \quad y'_{i'j'} = y_{\mu'(i',j')} \quad ((i', j') \in \mathcal{P}').$$

Define

$$\begin{aligned} v_{i1} &= \bar{y}_{i1} \otimes 1; \\ v_{ij} &= \bar{y}_{ij} \otimes \bar{y}'_{ij} \quad (1 \leq i \leq k; 2 \leq j \leq m_i - 1); \\ v_{im_i} &= \begin{cases} \bar{y}_{im_i} \otimes 1 & \text{if } m_i \text{ is even;} \\ \bar{y}_{im_i} \otimes \bar{y}'_{im_i} & \text{if } m_i \text{ is odd.} \end{cases} \end{aligned}$$

If m_i is even then, by Corollary 2.3,

$$\begin{aligned} [v_{i1}, v_{i2}, \dots, v_{im_i}] &= [\bar{y}_{i1}, \bar{y}_{i2}] [\bar{y}_{i3}, \bar{y}_{i4}] \dots [\bar{y}_{i(m_i-1)}, \bar{y}_{im_i}] \otimes [\bar{y}'_{i2}, \bar{y}'_{i3}] [\bar{y}'_{i4}, \bar{y}'_{i5}] \dots [\bar{y}'_{i(m_i-2)}, \bar{y}'_{i(m_i-1)}] \\ &= \bar{y}_{i1} \bar{y}_{i2} \bar{y}_{i3} \dots \bar{y}_{im_i} ((\bar{y}_{i1}, \bar{y}_{i2}) + 1) ((\bar{y}_{i3}, \bar{y}_{i4}) + 1) \dots ((\bar{y}_{i(m_i-1)}, \bar{y}_{im_i}) + 1) \\ &\quad \otimes \bar{y}'_{i2} \bar{y}'_{i3} \dots \bar{y}'_{i(m_i-1)} ((\bar{y}'_{i2}, \bar{y}'_{i3}) + 1) ((\bar{y}'_{i4}, \bar{y}'_{i5}) + 1) \dots ((\bar{y}'_{i(m_i-2)}, \bar{y}'_{i(m_i-1)}) + 1) \end{aligned}$$

If m_i is odd then, by the same corollary,

$$\begin{aligned} [v_{i1}, v_{i2}, \dots, v_{im_i}] &= [\bar{y}_{i1}, \bar{y}_{i2}] [\bar{y}_{i3}, \bar{y}_{i4}] \dots [\bar{y}_{i(m_i-2)}, \bar{y}_{i(m_i-1)}] \bar{y}_{im_i} \otimes [\bar{y}'_{i2}, \bar{y}'_{i3}] [\bar{y}'_{i4}, \bar{y}'_{i5}] \dots [\bar{y}'_{i(m_i-1)}, \bar{y}'_{im_i}] \\ &= \bar{y}_{i1} \bar{y}_{i2} \bar{y}_{i3} \dots \bar{y}_{i(m_i-1)} \bar{y}_{im_i} ((\bar{y}_{i1}, \bar{y}_{i2}) + 1) ((\bar{y}_{i3}, \bar{y}_{i4}) + 1) \dots ((\bar{y}_{i(m_i-2)}, \bar{y}_{i(m_i-1)}) + 1) \\ &\quad \otimes \bar{y}'_{i2} \bar{y}'_{i3} \dots \bar{y}'_{i(m_i-1)} \bar{y}'_{im_i} ((\bar{y}'_{i2}, \bar{y}'_{i3}) + 1) ((\bar{y}'_{i4}, \bar{y}'_{i5}) + 1) \dots ((\bar{y}'_{i(m_i-1)}, \bar{y}'_{im_i}) + 1) \end{aligned}$$

It follows that

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] = \bar{y} \ Q \otimes \bar{y}' \ Q'$$

where

$$\begin{aligned} \bar{y} &= \prod_{i=1}^k \prod_{j=1}^{m_i} \bar{y}_{ij}, & \bar{y}' &= \prod_{i'=1}^k \prod_{j'=2}^{m'_{i'}} \bar{y}'_{i'j'}, \\ m'_{i'} &= \begin{cases} m_{i'} - 1 & \text{if } m_{i'} \text{ is even;} \\ m_{i'} & \text{if } m_{i'} \text{ is odd,} \end{cases} \\ Q &= \prod_{i=1}^k \prod_{j=1}^{\lfloor \frac{m_i}{2} \rfloor} ((\bar{y}_{i(2j-1)}, \bar{y}_{i(2j)}) + 1), & Q' &= \prod_{i'=1}^k \prod_{j'=1}^{\lfloor \frac{m'_{i'}-1}{2} \rfloor} ((\bar{y}'_{i'(2j')}, \bar{y}'_{i'(2j'+1)}) + 1). \end{aligned}$$

Since μ is injective, all elements $y_{i(2j-1)}, y_{i(2j)}$ ($i = 1, 2, \dots, k; j = 1, 2, \dots, \lfloor \frac{m_i}{2} \rfloor$) that appear in Q are distinct elements of the set $\{y_1, y_2, \dots\}$. Hence, by Corollary 2.7, we have $Q \neq 0$ in $G = F\mathcal{G}/I$. Similarly, $Q' \neq 0$ in $H = F\mathcal{G}_r/I_r$. Since \bar{y} and \bar{y}' are invertible elements of G and H , respectively, we have $\bar{y} \ Q \otimes \bar{y}' \ Q' \neq 0$, that is,

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] \neq 0,$$

as required.

This completes the proof of Theorem 1.9. □

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